

ANGLES IN C*-ALGEBRAS

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ABSTRACT. In this work we characterise the C*-algebras \mathcal{A} generated by projections with the property that every pair of projections in \mathcal{A} has positive angle, as certain extensions of abelian algebras by algebras of compact operators. We show that this property is equivalent to a lattice theoretic property of projections and also to the property that the set of finite-dimensional *-subalgebras of \mathcal{A} is directed.

1. INTRODUCTION

The structure of the set of projections of an operator algebra has been one of the main objects of investigation from the beginnings of the theory. In many cases, knowledge of the projections and their equivalence classes provides useful information about the structure of the algebra. The structure of projections in a C*-algebra is being studied extensively. One can find a survey of results in [3].

The C*-algebra generated by two projections is analysed in Pedersen [15] and in Raeburn and Sinclair [17]. The relative position of two Hilbert space projections is studied in Dixmier [7], Davis [6] and Halmos [10]. Some lattice-theoretic implications of positive angles appear in Mackey [14]. Also, consequences of this geometric property for the structure of \mathcal{A} may be found in Akemann [1]. Actually, this paper motivated some of our results in Section 3.

D. Topping [18] has characterised the von Neumann algebras \mathcal{M} with the property that every pair of projections in \mathcal{M} has positive angle. In this work we characterise the C*-algebras \mathcal{A} generated by projections and having this property, which we call *the positive angle property*.

In Section 2 we define the notion of angle between two projections in a C*-algebra \mathcal{A} and show that it does not depend on the (faithful) representation of \mathcal{A} .

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We say that a C^* -algebra \mathcal{A} has the *sublattice property* if the set $\mathcal{P}(\mathcal{A})$ of all projections in \mathcal{A} is a sublattice of the projection lattice of its enveloping von Neumann algebra \mathcal{A}^{**} . This property is studied in Section 3. The main result is Theorem 3.6 which asserts that the sublattice property is equivalent to the positive angle property.

In Section 4 we analyse the structure of a C^* -algebra which satisfies the positive angle property. Our analysis has two main consequences:

We show that the positive angle property is equivalent to the *directed set property*. A C^* -algebra \mathcal{A} is said to have this property [13] if the set of finite dimensional C^* -subalgebras of \mathcal{A} is directed by inclusion.

Thus a geometric, a lattice-theoretic and an algebraic property are shown to be equivalent.

The main result of our analysis concerns the structure of a C^* -algebra \mathcal{A} satisfying the positive angle property and generated by projections. Such an algebra is characterized (Theorem 4.6) as an extension $0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow 0$ where \mathcal{K} is a C^* -algebra of compact operators, \mathcal{C} is an abelian C^* -algebra generated by projections, and the Busby invariant of the extension takes values in the centre of the corona of \mathcal{K} . Alternatively, \mathcal{A} can be written as a sum $\mathcal{K} + \mathcal{Z}$, where \mathcal{Z} is its centre.

In particular, we characterise the AF algebras having the positive angle property, or equivalently the directed set property. Lazar in [13] has given a different characterisation of these algebras in spectral terms.

As a corollary, we show that Topping's characterisation of von Neumann algebras with the positive angle property extends to AW^* algebras.

Notation In general we follow the notation of [16]. By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators on a Hilbert space H and by $\mathcal{K}(H)$ the subalgebra of all compact operators on H .

If \mathcal{A} is a C^* -algebra we denote by $\mathcal{P}(\mathcal{A})$ the partially ordered set of all projections in \mathcal{A} . When \mathcal{A} is a von Neumann algebra (or more generally, an AW^* -algebra), then $\mathcal{P}(\mathcal{A})$ is a (complete) lattice. If p, q are projections, we denote by $p \vee q$ their supremum and by $p \wedge q$ their infimum; for a family \mathcal{E} we use the symbols $\vee \mathcal{E}$ and $\wedge \mathcal{E}$. The partially ordered set where these are calculated will be clear from the context.

The universal representation of \mathcal{A} is denoted by (π_u, H_u) . We often identify \mathcal{A} with $\pi_u(\mathcal{A})$, and the second dual \mathcal{A}^{**} with the bicommutant $\pi_u(\mathcal{A})''$. If X is a subset of a vector space we denote by $[X]$ the linear span of X . For a family $\{X_j\}$ of Banach spaces, we use the symbol $\sum_{j \in \mathcal{J}} \oplus_{\ell^\infty} X_j$ for the space of all bounded families (x_j) with $x_j \in X_j$ and the symbol $\sum_{j \in \mathcal{J}} \oplus_{c_0} X_j$ for the subspace of families (x_j) with $(\|x_j\|) \in c_0(\mathcal{J})$.

2. ANGLES

In this section we show that the notion of angle between two projections in a C*-algebra can be defined intrinsically; in particular, it is the same in any faithful representation of the algebra (see Remark 2.6).

If p, q are in a C*-algebra \mathcal{A} , we denote by $C^*(p, q)$ the C*-subalgebra of \mathcal{A} generated by p and q . We will use the following description of the C*-algebra generated by two projections due to Pedersen [15].

Proposition 2.1. *Let p, q be projections on a Hilbert space H and let $\sigma = \text{sp}(pqp)$, $\sigma_0 = \sigma \setminus \{0\}$, $\sigma_1 = \sigma \setminus \{1\}$, $\sigma_{01} = \sigma \setminus \{0, 1\}$. Then the C*-algebra $\mathcal{C} = C^*(p, q)$ is isomorphic to a subalgebra of $M_2(C(\sigma))$. More precisely, if \mathcal{C}_o is the C*-subalgebra of \mathcal{C} generated by pq then*

$$\begin{aligned} \mathcal{C}_o &\simeq \begin{bmatrix} C_o(\sigma_0) & C_o(\sigma_{01}) \\ C_o(\sigma_{01}) & C_o(\sigma_1) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} : f_{11}(0) = 0, f_{ij}(0) = 0 = f_{ij}(1) \text{ for } ij > 1 \right\}. \end{aligned}$$

(a) If $0 \in \overline{\sigma_0}$, then

$$\begin{aligned} \mathcal{C} &\simeq \begin{bmatrix} C(\sigma) & C_o(\sigma_{01}) \\ C_o(\sigma_{01}) & C_o(\sigma_1) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} : f_{ij}(0) = 0 = f_{ij}(1) \text{ for } i \neq j, f_{22}(1) = 0 \right\}. \end{aligned}$$

(b) If $0 \notin \overline{\sigma_0}$, then

$$\mathcal{C} = \mathcal{C}_o \oplus \mathbb{C}(p \wedge q^\perp) \oplus \mathbb{C}(p^\perp \wedge q).$$

The next proposition will be used repeatedly throughout the paper.

Proposition 2.2. *Let $p, q \in \mathcal{B}(H)$ be projections, $e = p \wedge q$ and $f = p \vee q$. The following are equivalent:*

- (a) *The sequence $\{(pqp)^n\}_n$ is norm-convergent (to e);*
- (b) *The point 1 is not an accumulation point of $\text{sp}(pqp)$;*
- (c) $\|(p - e)(q - e)\| < 1$;
- (d) $f \in C^*(p, q)$;
- (e) $C^*(p, q)$ is unital.

Proof. (a) \Rightarrow (b) It is well known that $p \wedge q$ is the strong limit of the sequence $\{(pqp)^n\}_n$ (in $\mathcal{B}(H)$). Thus if $(pqp)^n$ is norm-convergent, it converges to e .

Now pqp is a positive contraction, hence $C^*(pqp) \simeq C_o(\text{sp}(pqp) \setminus \{0\})$.

If $\phi \in C_o(\text{sp}(pqp) \setminus \{0\})$ denotes the function $\phi(t) = t$, then ϕ^n decreases pointwise to $\chi_{\{1\}}$ (the characteristic function of $\{1\}$). Thus if $(pqp)^n$ converges in norm, then ϕ^n converges uniformly and so $\chi_{\{1\}}$

must be continuous on $\text{sp}(pqp)$; hence 1 cannot be an accumulation point of $\text{sp}(pqp)$.

(b) \Rightarrow (c) If 1 is not an accumulation point of $\text{sp}(pqp)$, then, in the notation of the previous paragraph, $\|(1 - \chi_{\{1\}})\phi\|_\infty < 1$; thus $\|(1 - e)pqp\| < 1$ and so

$$\begin{aligned} \|(q - e)(p - e)\|^2 &= \|(p - e)(q - e)(p - e)\| = \|pqp - e\| \\ &= \|pqp(1 - e)\| < 1. \end{aligned}$$

(c) \Rightarrow (a) If $\|(q - e)(p - e)\| < 1$ then, as just observed, $(p - e)(q - e)(p - e)$ is a strict contraction and so $((p - e)(q - e)(p - e))^n$ is norm-convergent to 0; but it is easily verified that $((p - e)(q - e)(p - e))^n = (pqp)^n - e$.

(b) \Rightarrow (e) If 1 is not a limit point of $\text{sp}(pqp)$, then the characteristic function χ of σ_1 is in $C_0(\sigma_1)$. Thus if 0 is a limit point of $\sigma(pqp)$, then from part (a) of 2.1, the element

$$u = \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix},$$

is in the algebra and clearly acts as a unit. If 0 is not a limit point of $\sigma(pqp)$, then from part (b) of 2.1, the element

$$u = \begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix} \oplus p \wedge q^\perp \oplus p^\perp \wedge q,$$

where ψ is the characteristic function of $\sigma_{0,1}$ is in the algebra and acts as a unit for the algebra.

(e) \Rightarrow (b) We use Proposition 2.1: Let $u = [u_{ij}] \in M_2(C(\sigma))$ denote the unit of $\mathcal{C} \simeq C^*(p, q)$. If $g = [g_{ij}] \in \mathcal{C}$ is chosen so that $g_{12} = 0$, then the relation $ug = g$ yields $u_{22}g_{22} = g_{22}$. Since $g_{22} \in C_o(\sigma_1)$ is arbitrary, it follows that $u_{22}(t) = 1$ for all $t \in \sigma \setminus \{1\}$. Since $u_{22} \in C_o(\sigma_1)$, the point 1 cannot be an accumulation point of $\sigma = \text{sp}(pqp)$.

(e) \Leftrightarrow (d) If $f \in C^*(p, q)$, then since $fp = pf = p$ and $fq = qf = q$ it is clear that f is the unit of $C^*(p, q)$.

Conversely, let u be the unit of $C^*(p, q)$. Then $u \geq p$ and $u \geq q$, so $u \geq f$ (in $\mathcal{B}(H)$). Now u can be approximated in norm by polynomials ϕ_n in p and q . But since $fp = pf = p$ and $fq = qf = q$ we have $\phi_n f = f \phi_n = \phi_n$ for each n , so in the limit $fu = uf = u$. Thus $f \geq u$ and so $f = u$.

The proof is complete. \square

Remark 2.3. *The conditions of the proposition are also equivalent to*

(i) $\|(f - p)(f - q)\| < 1$
and to

(ii) The sequence $\{(p+q)^{1/n}\}_n$ is norm-convergent (to f).
 Since we will not use this fact, we omit the proof.

If π is a representation of a C^* -algebra \mathcal{A} , we denote by $\tilde{\pi}$ its unique extension to a normal (i.e. w^* -continuous) representation of \mathcal{A}^{**} .

Proposition 2.4. *Let p, q be projections in a C^* -algebra \mathcal{A} and $e = p \wedge q$, calculated in \mathcal{A}^{**} . If π is a faithful representation of \mathcal{A} on a Hilbert space H then*

- (i) $\tilde{\pi}(e)$ is the infimum of $\pi(p)$ and $\pi(q)$ calculated in $\mathcal{B}(H)$;
- (ii) $\|(\pi(p) - \tilde{\pi}(e))(\pi(q) - \tilde{\pi}(e))\| = \|(p - e)(q - e)\|$.

Proof. Let $e_1 = \pi(p) \wedge \pi(q)$. Since $e_1 = \lim_n (\pi(p)\pi(q)\pi(p))^n$ and $e = \lim_n (pqp)^n$ strongly, (i) follows from the normality of $\tilde{\pi}$.

Hence

$$\|(\pi(p) - e_1)(\pi(q) - e_1)\| = \|\tilde{\pi}((p - e)(q - e))\| \leq \|(p - e)(q - e)\| \leq 1.$$

If $\|(\pi(p) - e_1)(\pi(q) - e_1)\| = 1$ then we have equality throughout.

If on the other hand $\|(\pi(p) - e_1)(\pi(q) - e_1)\| < 1$ then, by Proposition 2.2, $e_1 = \tilde{\pi}(e)$ lies in the C^* -algebra $\pi(\mathcal{A})$. Thus there exists $r \in \mathcal{A}$ with $\tilde{\pi}(e) = \pi(r)$ and so

$$\|\pi(r) - (\pi(p)\pi(q)\pi(p))^n\| \rightarrow 0.$$

It follows that $\|r - (pqp)^n\| \rightarrow 0$. Since e is the strong operator limit of $(pqp)^n$, it must equal r ; thus $e \in \mathcal{A}$. But π is isometric, so

$$\|(\pi(p) - \tilde{\pi}(e))(\pi(q) - \tilde{\pi}(e))\| = \|(\pi(p) - \pi(e))(\pi(q) - \pi(e))\| = \|(p - e)(q - e)\|$$

as required. \square

Definition 2.5. *If \mathcal{A} is a C^* algebra and $p, q \in \mathcal{P}(\mathcal{A})$, we define*

$$c(p, q) = \|(p - p \wedge q)(q - p \wedge q)\|$$

*where $p \wedge q$ is calculated in \mathcal{A}^{**} .*

Remarks 2.6. (i) Let $p, q \in \mathcal{B}(H)$ be two projections with $pH \cap qH = 0$. Recall that the angle $\theta(p, q)$ between the subspaces pH and qH is defined to be the arc cosine of the quantity

$$\sup\{|\langle \xi, \eta \rangle| : \xi = p\xi, \eta = q\eta, \|\xi\|, \|\eta\| \leq 1\}.$$

Note that

$$\cos \theta(p, q) = \|pq\|.$$

It is well known that $\theta(p, q) > 0$ if and only if $pH + qH$ is closed.

(ii) It follows from Proposition 2.4 that if p and q are projections in a C^* -algebra \mathcal{A} ,

$$c(p, q) = \|(\pi(p) - \pi(p) \wedge \pi(q))(\pi(q) - \pi(p) \wedge \pi(q))\|$$

for any faithful representation π of \mathcal{A} . In particular, if \mathcal{A} is contained in a C*-algebra \mathcal{B} , then $c(p, q)$ is independent of whether $p \wedge q$ is calculated in \mathcal{A}^{**} or \mathcal{B}^{**} .

(iii) In Proposition 2.2 it is shown that $c(p, q) = 1$ if and only if 1 is an accumulation point of $\text{sp}(pqp)$. Applying the Proposition to the pair (p, q^\perp) we see that $c(p, q^\perp) = 1$ if and only if 0 is an accumulation point of $\text{sp}(pqp)$.

3. LATTICES OF PROJECTIONS

A C*-algebra \mathcal{A} is said to have the *directed set property* [13] if the set of finite dimensional C*-subalgebras of \mathcal{A} is directed by inclusion. It is said to have the *lattice property* [12] if the partially ordered set $\mathcal{P}(\mathcal{A})$ of projections in \mathcal{A} is a lattice in its own order. A. Lazar [12] has shown that these properties are equivalent for AF C*-algebras. In general, the lattice property does not imply the directed set property; consider for example the C*-algebra $\mathcal{B}(H)$.

We introduce a lattice-theoretic property which will prove to be equivalent to the directed set property: We say that \mathcal{A} has the *sublattice property* when $\mathcal{P}(\mathcal{A})$ is a sublattice of $\mathcal{P}(\mathcal{A}^{**})$ (of course $\mathcal{P}(\mathcal{A}^{**})$ is always a lattice, since \mathcal{A}^{**} is a von Neumann algebra).

The main result of this section is Theorem 3.6 which shows that a C*-algebra \mathcal{A} satisfies the sublattice property if and only if $c(p, q) < 1$ for each pair of projections $p, q \in \mathcal{A}$; this gives a geometric characterization of the algebras with the sublattice property. In Section 5 we will prove that this latter property is equivalent to the directed set property (Theorem 5.1).

If π is a faithful representation of a C*-algebra \mathcal{A} and $p, q \in \mathcal{A}$ are projections such that $\pi(p)$ and $\pi(q)$ satisfy the equivalent conditions of Proposition 2.2, then $\pi(p) \wedge \pi(q) \in C^*(\pi(p), \pi(q))$. The converse does not hold in general; however, it is true if π is the universal representation:

Proposition 3.1. *Let \mathcal{A} be a C*-algebra, $p, q \in \mathcal{A}$ be projections and $f = p \vee q$, $e = p \wedge q$ calculated in \mathcal{A}^{**} . The following are equivalent:*

- (i) $e \in \mathcal{A}$
- (ii) $c(p, q) < 1$
- (iii) $f \in \mathcal{A}$.

Proof. Proposition 2.2 shows that (iii) is equivalent to (ii) and (ii) implies (i).

(i) \Rightarrow (ii) Assume that $c(p, q) = 1$ and that $e \in \mathcal{A}$. Then the projections $p_o = p - e$ and $q_o = q - e$ are in \mathcal{A} . Now $c(p_o, q_o) = 1$

and hence 1 is an accumulation point of $\text{sp}(p_o q_o p_o)$, by Proposition 2.2. The C*-algebra generated by $\{p_o q_o p_o, p_o\}$ consists of continuous functions on $\text{sp}(p_o q_o p_o)$; let φ be the state of this algebra corresponding to evaluation at 1. Now define a state ψ of the C*-algebra generated by $\{p_o, q_o\}$ by $\psi(x) = \varphi(p_o x p_o)$, and extend it to a state $\tilde{\psi}$ of \mathcal{A} . By hypothesis, \mathcal{A} is sitting in its universal representation, hence $\tilde{\psi}$ is a vector state. Since $\tilde{\psi}(p_o) = \tilde{\psi}(q_o) = 1$, it follows that $\tilde{\psi}(p_o \wedge q_o) = 1$; this is impossible, since $p_o \wedge q_o = 0$. \square

Let \mathcal{A} be a C*-algebra. Recall [1] that a projection $p \in \mathcal{A}^{**}$ is said to be *open* if there exists an increasing net $a_i \in \mathcal{A}^+$ such that $a_i \nearrow p$. A projection $p \in \mathcal{A}^{**}$ is said to be *closed* if $1 - p$ is open. In case \mathcal{A} is unital, this means that p is the infimum of a decreasing net of contractions from \mathcal{A} .

Akemann [1, Theorem II.7] proves that if p, q are closed projections in \mathcal{A}^{**} and $c(p, q) < 1$, then $p \vee q$ is closed. The converse is not true in general, as we show below. However, the following is immediate from Proposition 3.1.

Corollary 3.2. *Let \mathcal{A} be a unital C*-algebra, $p, q \in \mathcal{A}$ be projections and $f = p \vee q$ calculated in \mathcal{A}^{**} . The projection f is closed if and only if $c(p, q) < 1$.*

Proof. The supremum of a set of projections in \mathcal{A} is always open [1, Proposition II.5]. So, if f is closed then it must belong to \mathcal{A} [1, Proposition II.18]. By Proposition 3.1, this implies $c(p, q) < 1$. \square

The following proposition shows that this Corollary does not hold if we do not assume that $p, q \in \mathcal{A}$.

Proposition 3.3. *There exists a unital C*-algebra \mathcal{A} and open projections p and q in \mathcal{A}^{**} such that $c(p, q) = 1$ and $p \wedge q$ is open.*

Proof. Let $\mathcal{A} = \mathbb{C}I + \mathcal{K}(H)$. If $\{\xi_n\}_{n \in \mathbb{N}}$ is orthonormal in H , consider the following compact operators

$$a = \sum_{n=1}^{\infty} \frac{1}{n} e_n, \quad b = \sum_{n=1}^{\infty} \frac{1}{n} f_n,$$

where e_n, f_n are the rank one projections with ranges spanned by ξ_{2n} and $\eta_n = \cos(\frac{1}{n})\xi_{2n} + \sin(\frac{1}{n})\xi_{2n+1}$, respectively. Let p be the strong limit in \mathcal{A}^{**} of the sequence $a^{1/k}$ and q the strong limit of the sequence $b^{1/k}$. Then p and q are open projections by construction and it is clear that $c(p, q) = 1$. We claim that $p \wedge q = 0$ and hence $p \wedge q$ is open. Indeed p and q are the range projections of $\pi_u(a)$ and $\pi_u(b)$ respectively. These

operators are ampliations of a and b [2, Corollary 1 of Theorem 1.4.4] hence the closures of their ranges do not intersect. \square

The following lemma will be needed in the proof of Theorem 3.6.

Lemma 3.4. *Let p and q be projections on a Hilbert space H and \mathcal{A} be the C^* -algebra generated by p and q . If $\text{sp}(pqp)$ is infinite then \mathcal{A} contains two distinct projections r and s with $c(s, r) = 1$.*

Proof. If $c(p, q) = 1$ we are done. Assume that $c(p, q) < 1$. This implies that \mathcal{A} is unital and that 1 is not an accumulation point of $\text{sp}(pqp)$ (Proposition 2.2). If 0 is an accumulation point of $\text{sp}(pqp)$, then $c(p^\perp, q) = 1$ and again we are done. Thus we may assume that $K = \text{sp}(pqp) \setminus \{0, 1\}$ is a compact subset of $(0, 1)$.

Since $\text{sp}(pqp)$ is infinite, it has an accumulation point $\delta \in K$. Using the description of the algebra \mathcal{A} given in Pedersen's theorem (Proposition 2.1), we construct projections $r, s \in \mathcal{A}$ with $c(s, r) = 1$. Since 0 is not an accumulation point of $\sigma = \text{sp}(pqp)$, the characteristic function χ of σ_0 is in $C_0(\sigma_0)$; thus by part (b) of Pedersen's theorem, the projection $s = \begin{pmatrix} \chi & 0 \\ 0 & 0 \end{pmatrix}$ is in \mathcal{A} .

Choose a continuous function $f : \text{sp}(pqp) \rightarrow [0, 1]$ supported in K , with $f(\delta) = 1$ and such that $f(\delta)$ is an accumulation point of $f(K)$. Define g on $\text{sp}(pqp)$ by $g(t) = (1 - f^2(t))^{\frac{1}{2}}$ for $t \in K$ and $g(t) = 0$ for $t \in \text{sp}(pqp) \setminus K$. Now consider $r = \begin{pmatrix} f^2 & fg \\ fg & g^2 \end{pmatrix}$. This is a projection in \mathcal{A} . Note that $srs = \begin{pmatrix} f^2 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that 1 is an accumulation point of $\text{sp}(srs)$ and so $c(s, r) = 1$. \square

Remark 3.5. *Consider the C^* -algebra $C^*(p, q)$ generated by two projections. If $c(r, s) < 1$ for all projections $r, s \in C^*(p, q)$, then the algebra is finite dimensional. Indeed, $\text{sp}(pqp)$ is finite by Lemma 3.4, and the claim follows from Proposition 2.1.*

We will prove in Theorem 5.2 that the same holds for the C^ -algebra generated by any finite number of projections.*

Theorem 3.6. *Let \mathcal{A} be a C^* -algebra. The following are equivalent:*

- (i) *The partially ordered set $\mathcal{P}(\mathcal{A})$ is a sublattice of the lattice $\mathcal{P}(\mathcal{A}^{**})$;*
- (ii) *$c(p, q) < 1$ for each pair $p, q \in \mathcal{A}$ of projections;*
- (iii) *$\text{sp}(pqp)$ is finite, for each pair $p, q \in \mathcal{A}$ of projections.*

Proof. The implications (ii) \Rightarrow (i) and (iii) \Rightarrow (ii) follow from Proposition 2.2 and (i) \Rightarrow (ii) follows from Proposition 3.1.

We prove (ii) \Rightarrow (iii): If $\text{sp}(pqp)$ is infinite for some projections $p, q \in \mathcal{A}$, then Lemma 3.4 shows that there exist projections $p_1, q_1 \in \mathcal{A}$ such that $c(p_1, q_1) = 1$. \square

4. THE POSITIVE ANGLE PROPERTY

We will say that a C*-algebra \mathcal{A} has *the positive angle property* when $c(p, q) < 1$ for all $p, q \in \mathcal{P}(\mathcal{A})$. Two examples are immediate: abelian C*-algebras and the algebra of all compact operators, or subalgebras of these. Of course, projectionless C*-algebras have the property in a trivial way, so we will be considering algebras generated by their projections. We will show that the most general C*-algebra with these two properties can be constructed from an abelian C*-algebra and a C*-algebra of compacts in the way described in Theorem 4.6.

Thus in this section, and up to Proposition 4.5, *we will let \mathcal{A} be a C*-algebra with the positive angle property which is generated, as a C*-algebra, by its projections.*

Note that if p, q are projections in \mathcal{A} then $C^*(p, q)$ is finite-dimensional (Remark 3.5) and hence w^* -closed in \mathcal{A}^{**} .

It will be convenient to identify \mathcal{A} with its reduced atomic representation acting on a Hilbert space H . Thus if $\mathcal{I} = \widehat{\mathcal{A}}$ is the spectrum of \mathcal{A} , and a representative π_i is chosen from each unitary equivalence class $i \in \mathcal{I}$, we have

$$\begin{aligned} \text{id} &= \sum_{j \in \mathcal{I}} \oplus (\pi_j, H_j) \\ \text{and } \mathcal{A}'' &= \sum_{j \in \mathcal{I}} \oplus \mathcal{B}(H_j) \end{aligned}$$

(see for example [11, 10.3.10].)

Denote by $z_j \in \mathcal{B}(H)$ the projection onto H_j . The z_j are pairwise orthogonal central projections in \mathcal{A}'' . Also, $\pi_j(a) = az_j$ for each $a \in \mathcal{A}$ and $j \in \mathcal{I}$.

Denote by $\mathcal{K}(\mathcal{A})$ the *compact elements* of \mathcal{A} , that is, the elements $a \in \mathcal{A}$ for which the operator $x \rightarrow axa$ is compact on \mathcal{A} . It follows from [20] and [9, Theorem 3.7] that $\mathcal{K}(\mathcal{A}) = \mathcal{K}(H) \cap \mathcal{A}$. We will prove in Lemma 4.2 below that, if \mathcal{A} is not abelian, the ideal $\mathcal{K}(\mathcal{A})$ is nonzero. Let $\mathcal{J} = \{j \in \mathcal{I} : \pi_j(\mathcal{K}(\mathcal{A})) \neq 0\}$. Note that $j \in \mathcal{J}$ iff $\mathcal{K}(\mathcal{A})z_j \neq 0$.

From [8, 2.11.2], the spectrum of $\mathcal{K}(\mathcal{A})$ is in bijective correspondence with \mathcal{J} . It now follows by standard arguments (see eg. [2, 1.4.5]) that

$$\mathcal{K}(\mathcal{A}) = \sum_{j \in \mathcal{J}} \oplus_{c_o} \mathcal{K}(\mathcal{A})z_j \oplus 0 = \sum_{j \in \mathcal{J}} \oplus_{c_o} \mathcal{K}_j \oplus 0$$

where each $\mathcal{K}_j = \mathcal{K}(\mathcal{A})z_j$ is an ideal of \mathcal{A} and is equal to $\mathcal{K}(H_j)$.

Remark 4.1. *The centre $\mathcal{Z}(\mathcal{A})$ acts ‘componentwise’ on $\mathcal{K}(\mathcal{A})$: if $a \in \mathcal{Z}(\mathcal{A})$ and $j \in \mathcal{J}$ there is a scalar $\lambda_j(a)$ such that*

$$az_j = \lambda_j(a)z_j.$$

Also, the intersection $\mathcal{K}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A})$ is isomorphic to $c_o(\mathcal{J}_f)$ where $\mathcal{J}_f = \{j \in \mathcal{J} : \dim H_j < \infty\}$.

Indeed, since $\mathcal{K}_j \subseteq \mathcal{A}$ it follows that a commutes with \mathcal{K}_j ; hence so does az_j . Since $\mathcal{K}_j = \mathcal{K}(H_j)$, az_j is a scalar multiple of the identity z_j of H_j .

The second claim is obvious.

Lemma 4.2. *If $f \in \mathcal{A}$ is a minimal projection then $f\mathcal{A}f = \mathbb{C}f$. Thus, the minimal projections in \mathcal{A} are precisely the rank one projections of \mathcal{K}_j , $j \in \mathcal{J}$.*

Proof. First observe that if $p, q \in \mathcal{P}(\mathcal{A})$, then $C^*(p, q)$ is finite dimensional, hence is linearly generated by its projections. Thus

$$pq = \sum_{i=1}^n \lambda_i r_i, \quad r_i \in \mathcal{P}(\mathcal{A}), \quad \lambda_i \in \mathbb{C}.$$

An obvious induction shows that any word $w = p_1 p_2 \dots p_n$ can be written

$$w = \sum_{i=1}^n \lambda_i g_i, \quad g_i \in \mathcal{P}(\mathcal{A}) \quad \text{and so} \quad f w f = \sum_{i=1}^n \lambda_i f g_i f.$$

Since each $C^*(f, g_i)$ is finite dimensional, by spectral theory $f g_i f$ is a linear combination of projections in \mathcal{A} , necessarily no larger than f . Since f is minimal, each $f g_i f$ is therefore a scalar multiple of f , hence so is $f w f$. Since \mathcal{A} is generated by $\mathcal{P}(\mathcal{A})$, the first claim is proved.

Note that the minimal projections of \mathcal{K}_j are clearly minimal in \mathcal{A} . Conversely, by the previous paragraph, every minimal projection p of \mathcal{A} is compact and hence belongs to some \mathcal{K}_j . Clearly, p must be of rank one. \square

Recall that two projections $p, q \in \mathcal{P}(\mathcal{A})$ are said to be (Murray - von Neumann) equivalent if there exists a partial isometry $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$; we write $p \sim q$.

Proposition 4.3. *If $p \in \mathcal{P}(\mathcal{A})$, there does not exist an infinite orthogonal family $\{e_n, f_n\}_{n=1}^\infty \subseteq \mathcal{A}$ of nonzero projections in \mathcal{A} such that $e_n \leq p$, $f_n \leq p^\perp$ and $e_n \sim f_n$, for each $n \in \mathbb{N}$.*

Proof. Suppose such a family exists. We will reach a contradiction by constructing a projection $q \in \mathcal{A}$ such that $c(p, q) = 1$.

The sum $\sum_k p e_k$ converges strongly to a projection in \mathcal{A}'' . Define $p_1 = p - \sum_k p e_k$, so that

$$p = \sum_k p e_k + p_1 = \sum_k e_k + p_1$$

since $e_k \leq p$. If $v_k v_k^* = e_k$ and $v_k^* v_k = f_k$, set $q_k = c_k^2 e_k + c_k s_k (v_k + v_k^*) + s_k^2 f_k$ where c_k, s_k are real scalars with $c_k^2 + s_k^2 = 1$ and $c_k \rightarrow 1$. The q_k are pairwise orthogonal projections and are all orthogonal to p_1 (because $p_1 \leq p$ while $f_k \leq p^\perp$ so $p_1 \perp f_k$, and $p_1 \perp e_k$ by construction). Define

$$q = \sum_k q_k + p_1.$$

We show that $q \in \mathcal{A}$. We have

$$q = \sum_{k=1}^{\infty} q_k + p_1 = \left(\sum_{k=1}^{\infty} e_k + p_1 \right) + \sum_{k=1}^{\infty} (q_k - e_k)$$

and the first term is in \mathcal{A} (it is equal to p) while the second converges in norm.

Indeed, it is in fact norm Cauchy, since

$$\begin{aligned} \|e_k - q_k\| &= \|(1 - c_k^2)e_k - c_k s_k (v_k + v_k^*) - s_k^2 f_k\| \\ &= s_k \|s_k e_k - c_k (v_k + v_k^*) - s_k f_k\| \leq 4s_k \rightarrow 0 \end{aligned}$$

and

$$\left\| \sum_{k=m}^n (q_k - e_k) \right\| = \max_{m \leq k \leq n} \|q_k - e_k\|.$$

Setting $g_k = e_k + f_k$, it is easy to see that $g_k p = p g_k = e_k$ and $g_k q = q g_k = q_k$ for all k . Hence

$$(p \wedge q) g_k = g_k (p \wedge q) = (p g_k) \wedge (q g_k) = e_k \wedge q_k = 0.$$

Thus, $p \wedge q = p_1$. It follows that $c(e_k, q_k) = \|e_k q_k\| \leq \|p q - p_1\| = c(p, q)$ for each k and so $c(p, q) = 1$. \square

Lemma 4.4. *If $p \in \mathcal{P}(\mathcal{A})$ is not central then there exist minimal projections $e, f \in \mathcal{P}(\mathcal{A})$ such that $e \leq p$, $f \leq p^\perp$ and $e \sim f$.*

Proof. First, we claim that there exist non-zero projections $e_0, f_0 \in \mathcal{A}$ such that $e_0 \leq p$, $f_0 \leq p^\perp$ and $e_0 \sim f_0$.

Indeed, since $\mathcal{P}(\mathcal{A})$ generates \mathcal{A} , there exists $r \in \mathcal{P}(\mathcal{A})$ with $p r p^\perp \neq 0$. Let $p r p^\perp = v |p r p^\perp|$ be the corresponding polar decomposition. Then v belongs to $C^*(p, r)$. Set $e_0 = v v^* \leq p$ and $f_0 = v^* v \leq p^\perp$.

Now suppose that e_0 does not majorise a minimal projection. Then there is an infinite strictly decreasing sequence (q_n) of proper nonzero subprojections of e_0 ; set $e_1 = e_0 - q_1$ and $e_n = q_{n-1} - q_n$ for $n \geq 2$. If $f_n = v^*e_nv$, the family $\{e_n, f_n\}$ is orthogonal and satisfies $e_n \leq e_0$, $f_n \leq e_0^\perp$ and $e_n \sim f_n$ for all n . By Proposition 4.3, this is a contradiction.

Thus there exists a minimal subprojection e of e_0 ; set $f = v^*ev$. \square

Proposition 4.5. *Each projection in \mathcal{A} is a linear combination of central and finite rank projections. In fact $\mathcal{P}(\mathcal{A}) \subseteq [\mathcal{P}(\mathcal{K}(\mathcal{A}))] + \mathcal{P}(\mathcal{Z}(\mathcal{A}))$.*

Proof. Let $p \in \mathcal{P}(\mathcal{A})$. We claim that the set

$$\mathcal{J}_0 = \{j \in \mathcal{J} : 0 < pz_j < z_j\}$$

is finite. Indeed, for each $j \in \mathcal{J}_0$, choose projections e_j, f_j of rank one with $e_j \leq pz_j$ and $f_j \leq z_j - pz_j$. If \mathcal{J}_0 is infinite, we obtain an infinite family $\{e_j, f_j : j \in \mathcal{J}_0\} \subseteq \mathcal{A}$ which violates Proposition 4.3.

Note that, for each j , the projections pz_j and $p^\perp z_j$ cannot both have infinite rank. Indeed, if they did, since the C^* -algebra $\mathcal{A}z_j$ contains $\mathcal{K}(H_j)$, there would be rank one projections $\{e_n, f_n, n \in \mathbb{N}\}$ in $\mathcal{A}z_j$ with $e_n \leq p$ and $f_n \leq p^\perp$. As in Proposition 4.3, this contradicts the positive angle property.

Set $\mathcal{J}_1 = \{j \in \mathcal{J}_0 : \dim p H_j < \infty\}$ and $\mathcal{J}_2 = \mathcal{J}_0 \setminus \mathcal{J}_1$. If $j \in \mathcal{J}_1$ (resp. $j \in \mathcal{J}_2$), then the projection pz_j (resp. $p^\perp z_j$) has finite rank hence is in \mathcal{A} . Put $p_1 = \sum_{j \in \mathcal{J}_1} pz_j$ and $p_2 = \sum_{j \in \mathcal{J}_2} p^\perp z_j$. These are finite orthogonal sums of finite rank projections in \mathcal{A} , hence belong to $\mathcal{K}(\mathcal{A})$. Since $p - p_1$ and p_2 are orthogonal projections, the sum $q = p - p_1 + p_2$ is in $\mathcal{P}(\mathcal{A})$.

We claim that $q \in \mathcal{Z}(\mathcal{A})$. If not then Lemma 4.4 yields the existence of minimal equivalent projections $e, f \in \mathcal{A}$ such that $e \leq q$ and $f \leq q^\perp$. By Lemma 4.2, there exists j such that $e \leq z_j$ and $f \leq z_j$. But by construction qz_j is equal to either zero or z_j , a contradiction. \square

In what follows we will need the notion of the *Busby invariant*, introduced in [4]. Recall (see for instance [19, Section 3.2]) that if \mathcal{K} is an ideal of a C^* algebra \mathcal{A} and \mathcal{C} is the quotient \mathcal{A}/\mathcal{K} , the *Busby invariant* of the extension $0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow 0$ is the morphism $\tau : \mathcal{C} \rightarrow M(\mathcal{K})/\mathcal{K}$ defined as follows: Let $\sigma : \mathcal{A} \rightarrow M(\mathcal{K})$ be the unique extension of the inclusion map $\mathcal{K} \rightarrow M(\mathcal{K})$. Given $c \in \mathcal{C}$ let $a \in \mathcal{A}$ be any lift of c ; then one sets $\tau(c) = \pi(\sigma(a))$, where $\pi : M(\mathcal{K}) \rightarrow M(\mathcal{K})/\mathcal{K}$ is the quotient map. We will use this notation below.

Theorem 4.6. *Let \mathcal{A} be a C^* -algebra. The following are equivalent:*

- (i) \mathcal{A} has the positive angle property and is generated by its projections;

- (ii) $\mathcal{A} = \mathcal{K}(\mathcal{A}) + \mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{A})$ is generated by its projections;
 (iii) \mathcal{A} is an extension of \mathcal{C} by \mathcal{K} , where \mathcal{C} is an abelian C*-algebra generated by its projections, \mathcal{K} is a C*-algebra of compact operators, and the Busby invariant $\tau : \mathcal{C} \rightarrow M(\mathcal{K})/\mathcal{K}$ takes values in the centre of $M(\mathcal{K})/\mathcal{K}$.

Proof. (i) \Rightarrow (ii) If $p \in \mathcal{P}(\mathcal{A})$ then $p \in [\mathcal{P}(\mathcal{K}(\mathcal{A}))] + \mathcal{P}(\mathcal{Z}(\mathcal{A}))$ (Proposition 4.5). Since $\mathcal{K}(\mathcal{A}) + C^*(\mathcal{P}(\mathcal{Z}(\mathcal{A})))$ is a C*-subalgebra of \mathcal{A} and the latter is generated by its projections, we have

$$\mathcal{A} \subseteq \mathcal{K}(\mathcal{A}) + C^*(\mathcal{P}(\mathcal{Z}(\mathcal{A}))) \subseteq \mathcal{K}(\mathcal{A}) + \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$$

and hence equality holds. Since $\mathcal{A} = \mathcal{K}(\mathcal{A}) + C^*(\mathcal{P}(\mathcal{Z}(\mathcal{A})))$, it follows that

$$\mathcal{Z}(\mathcal{A}) = (\mathcal{Z}(\mathcal{A}) \cap \mathcal{K}(\mathcal{A})) + C^*(\mathcal{P}(\mathcal{Z}(\mathcal{A}))).$$

But $\mathcal{Z}(\mathcal{A}) \cap \mathcal{K}(\mathcal{A}) \simeq c_o(\mathcal{J}_f)$ (Remark 4.1) and so $\mathcal{Z}(\mathcal{A})$ is generated by its projections.

(ii) \Rightarrow (iii) We may assume that $\mathcal{K} = \mathcal{K}(\mathcal{A}) = \sum_{j \in \mathcal{J}} \oplus_{c_o} \mathcal{K}(H_j)$ for some Hilbert spaces H_j . Let $H_c = \bigvee_{j \in \mathcal{J}} H_j$.

Define $\mathcal{C} = \mathcal{A}/\mathcal{K}$ and note that \mathcal{C} is generated by its projections. The morphism $\tau : \mathcal{C} \rightarrow M(\mathcal{K})/\mathcal{K}$ can be obtained in the following way: Let $c \in \mathcal{C}$ and $a \in \mathcal{A}$ be any lift of c . Then $\tau(c) = \pi(a|_{H_c})$. Indeed, since \mathcal{K} acts non-degenerately on H_c , we have that $a|_{H_c}$ is an element of the multiplier algebra $M(\mathcal{K})$ and the map $a \rightarrow a|_{H_c}$ is a morphism extending the inclusion $\mathcal{K} \rightarrow M(\mathcal{K})$, hence coincides with σ . Writing $a = b + z$ with $b \in \mathcal{K}$ and $z \in \mathcal{Z}(\mathcal{A})$, we see that $\tau(c) = \pi((b + z)|_{H_c}) = \pi(z|_{H_c})$. Since $\mathcal{Z}(\mathcal{A})$ acts ‘componentwise’ on $\mathcal{K}(\mathcal{A})$, it follows that $z|_{H_c}$ belongs to the centre of $M(\mathcal{K})$ and so $\tau(c)$ belongs to the centre of $M(\mathcal{K})/\mathcal{K}$.

(iii) \Rightarrow (i) Suppose that

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow 0$$

is an exact sequence, \mathcal{C} is abelian and generated by its projections, $\mathcal{K} = \sum_{j \in \mathcal{J}} \oplus_{c_o} \mathcal{K}(H_j)$ and the corresponding morphism $\tau : \mathcal{C} \rightarrow M(\mathcal{K})/\mathcal{K}$ takes values in the centre of $M(\mathcal{K})/\mathcal{K}$. The projection onto H_j is denoted z_j .

Recall that up to strong isomorphism of extensions [19, 3.2.11 and 3.2.13] we may write

$$\mathcal{A} = \{(a, c) \in M(\mathcal{K}) \oplus \mathcal{C} : \pi(a) = \tau(c)\}.$$

We first show that \mathcal{A} has the positive angle property.

Any projection $P \in \mathcal{A}$ has the form $P = (p, q)$ where $p \in M(\mathcal{K})$ and $q \in \mathcal{C}$ are projections. Since $\pi(p) = \tau(q) \in \mathcal{Z}((M(\mathcal{K})/\mathcal{K}))$,

writing $p = \oplus_{j \in \mathcal{J}} p_j$ with each $p_j \in M(\mathcal{K}_j) = \mathcal{B}(H_j)$, we have that $\oplus_{j \in \mathcal{J}} (p_j x_j - x_j p_j) \in \mathcal{K}$, for all $\oplus_{j \in \mathcal{J}} x_j \in M(\mathcal{K})$. Now if $0 < p_j < z_j$, there exists $x_j \in \mathcal{B}(H_j)$ of norm one such that $x_j = (z_j - p_j)x_j p_j$ and then $\|p_j x_j - x_j p_j\| = \|x_j\| = 1$. It follows that the set $\mathcal{J}_0 = \{j \in \mathcal{J} : 0 < p_j < z_j\}$ is finite. For each $j \in \mathcal{J}_0$, we easily verify that p_j commutes with $\mathcal{B}(H_j)$ modulo $\mathcal{K}(H_j)$; since the Calkin algebra has trivial centre [5, Theorem 2.9], this shows that $p_j = \lambda_j z_j + a_j$ where $\lambda_j \in \mathbb{C}$ and $a_j \in \mathcal{K}_j$. Hence p_j must have either finite rank or finite co-rank.

Take two projections $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in \mathcal{A} . By the previous paragraph, $c(p_1, p_2) < 1$; since \mathcal{C} is abelian, $c(q_1, q_2) = 0$. Thus $c(P_1, P_2) = \max\{c(p_1, p_2), c(q_1, q_2)\} < 1$ as required.

We now show that \mathcal{A} is generated by its projections.

Let \mathcal{A}' be the C^* -subalgebra of \mathcal{A} generated by the projections of \mathcal{A} . Clearly \mathcal{A}' contains \mathcal{K} . Since \mathcal{C} is generated by its projections, it suffices to show that every projection in \mathcal{C} lifts to a projection in \mathcal{A} , hence in \mathcal{A}' . Indeed, it will then follow that $\mathcal{A}'/\mathcal{K} = \mathcal{C} = \mathcal{A}/\mathcal{K}$ and therefore $\mathcal{A} = \mathcal{A}'$.

Thus let $q \in \mathcal{C}$ be a projection. Noting that $\tau(q) \in M(\mathcal{K})/\mathcal{K}$, let $a \in M(\mathcal{K})$ be any selfadjoint lift of $\tau(q)$. Recall that $M(\mathcal{K}) = \sum_{j \in \mathcal{J}} \oplus_{\ell^\infty} \mathcal{B}(H_j) \subseteq \mathcal{B}(H)$. If $p \in \mathcal{B}(H)$ is the spectral projection of a corresponding to the complement of the interval $[-1/2, 1/2]$, Calkin [5, Theorem 2.4] proves that $a - p$ is a compact operator; since $M(\mathcal{K})$ is actually a von Neumann algebra containing a , it follows that $p \in M(\mathcal{K})$. Thus $a - p$ is a compact operator and is in $M(\mathcal{K})$, hence it must belong to \mathcal{K} . Now $\pi(p) = \pi(a) = \tau(q)$, and hence $(p, q) \in \mathcal{A}$; this is the required lift of q to \mathcal{A} . \square

Remark 4.7. *The sum $\mathcal{K}(\mathcal{A}) + \mathcal{Z}(\mathcal{A})$ in the Theorem need not be direct. In fact, $\mathcal{K}(\mathcal{A})$ need not be complemented in \mathcal{A} (example: $\mathcal{A} = \ell^\infty$). Also, $\mathcal{Z}(\mathcal{A})$ need not have a complement which is an ideal of \mathcal{A} (example: M_2).*

D. Topping has shown in [18] that if \mathcal{A} is a von Neumann algebra such that $c(p, q) < 1$ for each pair of projections $p, q \in \mathcal{A}$ then \mathcal{A} is the direct sum of a finite dimensional von Neumann algebra and an abelian one. As a corollary to Theorem 4.6 we obtain the following generalisation of Topping's result:

Proposition 4.8. *If \mathcal{A} is an AW^* -algebra with the positive angle property then \mathcal{A} is the direct sum of a finite dimensional AW^* -algebra and an abelian one.*

Proof. We use the notation introduced in the beginning of this section. Let $\mathcal{J}_1 = \{j \in \mathcal{J} : \dim H_j > 1\}$. For each $j \in \mathcal{J}_1$, the projection z_j

majorises at least two minimal orthogonal projections e_j, f_j in \mathcal{A} . The completeness of the lattice $\mathcal{P}(\mathcal{A})$ implies that $p = \vee e_j$ is in $\mathcal{P}(\mathcal{A})$. If \mathcal{J}_1 is infinite, this contradicts Proposition 4.3. Hence, \mathcal{J}_1 is finite.

If some z_j majorises an infinite sequence $\{e_n\}$ of pairwise orthogonal elements, the projection $p = \vee_n e_{2n}$ is in \mathcal{A} , which again contradicts Proposition 4.3. Thus H_j must be finite dimensional for each $j \in \mathcal{J}_1$.

Let $z = \sum_{j \in \mathcal{J}_1} z_j$. Then $z\mathcal{A}$ is finite-dimensional, $z^\perp\mathcal{A} \subseteq \mathcal{Z}(\mathcal{A})$ and \mathcal{A} is the direct sum of these two subalgebras. \square

5. EQUIVALENCE OF THE DIRECTED SET AND THE POSITIVE ANGLE PROPERTIES

Theorem 5.1. *A C*-algebra has the directed set property if and only if it has the positive angle property.*

Proof. If a C*-algebra has the directed set property, then, for each pair p, q of projections, the algebra $C^*(p, q)$ is clearly finite-dimensional, and hence $c(p, q) < 1$.

For the converse, since every finite-dimensional C*-algebra is generated by finitely many projections, it suffices to prove

Theorem 5.2. *Let \mathcal{A} be a C*-algebra with the positive angle property which is generated by finitely many projections. Then \mathcal{A} is finite-dimensional.*

Proof. Note that \mathcal{A} must be unital. We use the notation of Section 4; in particular, recall that the projections z_j commute with $\mathcal{A} \subseteq \mathcal{B}(H)$ and their sum is the identity operator on H .

Fix a finite generating set $\mathcal{F} \subseteq \mathcal{P}(\mathcal{A})$. Let $\mathcal{J}_0 \subseteq \mathcal{J}$ be the set of $j \in \mathcal{J}$ for which there exists $p \in \mathcal{F}$ such that $0 \neq pz_j \neq z_j$. It follows from the proof of Proposition 4.5 that \mathcal{J}_0 is finite.

Let $z = \sum_{j \in \mathcal{J}_0} z_j$. For each $p \in \mathcal{F}$ and each $j \in \mathcal{J}_1 = \mathcal{J} \setminus \mathcal{J}_0$, the projection pz_j is either 0 or z_j . Since

$$z^\perp p = \sum_{j \in \mathcal{J}} z^\perp pz_j = \sum_{j \in \mathcal{J}_1} z^\perp pz_j$$

we see that $z^\perp p$ is a (finite) sum of z_j 's, hence commutes with \mathcal{A} . In particular, the set $\{z^\perp p : p \in \mathcal{F}\}$ is commutative, hence so is the algebra $z^\perp\mathcal{A}$. Being generated by a finite number of projections, it is finite-dimensional.

Therefore it remains to show that the algebra $z\mathcal{A}$ is also finite dimensional. This follows from the

Claim For each $j \in \mathcal{J}$, the algebra $z_j\mathcal{A}$ is finite-dimensional.

Proof of the Claim. Since $\mathcal{A}z_j$ is equal to either $\mathcal{K}(H_j)$ or $\mathcal{K}(H_j) + \mathbb{C}z_j$ (by Theorem 4.6), each projection $p z_j \in \mathcal{A}z_j$ is either of finite rank or of finite co-rank.

Thus, $z_j \mathcal{A}$ is contained in a C^* -algebra generated by the unit z_j and finitely many projections p_1, \dots, p_k of finite rank. If H'_j is the Hilbert space generated by $p_i H_j, i = 1, \dots, k$, then H'_j is finite dimensional and $z_j \mathcal{A} \subseteq \mathcal{B}(H'_j) + \mathbb{C}z_j$, which is finite dimensional. \square

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